COMPLEX NUMBER

1. DEFINITION :

Complex numbers are defined as expressions of the form a + ib where a , b $\in$ R & $i = \sqrt{-1}$. It is denoted by z i.e. z = a + ib. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (Im z).

Every Complex Number Can Be Regarded As

Purely real	Purely imaginary	Imaginary
if b = 0	if a = 0	if b $\neq$ 0

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i = -1$; $i^3 = -i$; $i^4 = 1$ etc.

In general $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where $n \in I$

(iv) $\sqrt{a}\,\sqrt{b} = \sqrt{a\,b}$ only if atleast one of either a or b is non- negative.

Illustration 1 : The value of $i^{57} + 1/i^{125}$ is :-

(A) 0 (B) $-2i$ (C) 2i (D) 2

Solution :

$$i^{57} + 1/i^{125} = i^{56}.\ i + \frac{1}{i^{124}.i}$$

$$= \left(i^4\right)^{14} i + \frac{1}{\left(i^4\right)^{31} i}$$

$$= i + \frac{1}{i} = i + \frac{i}{i^2} = i - i = 0$$

Ans. (A)

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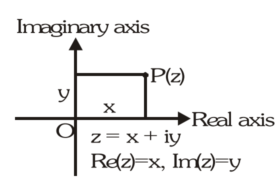
2. ARGAND DIAGRAM :

Master Argand had done a systematic study on complex numbers and represented every complex number z = x + iy as a set of ordered pair (x, y) on a plane called complex plane (Argand Diagram) containing two perpendicular axes. Horizontal axis is known as Real axis & vertical axis is known as Imaginary axis.

All complex numbers lying on the real axis are called as purely real and those lying on imaginary axis as purely imaginary.



3. ALGEBRAIC OPERATIONS :

Fundamental operations with complex numbers :

(a) Addition $(a + bi) + (c + di) = (a + c) + (b + d)i$

(b) Subtraction $(a + bi) - (c + di) = (a - c) + (b - d)i$

(c) Multiplication $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

(d) Division $\frac{a+bi}{c+di} = \frac{a+bi}{c+di}.\frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$

Note :

(i) The algebraic operations on complex numbers are similar to those on real numbers treating i as a polynomial.

(ii) Inequalities in complex numbers (non-real) are not defined. There is no validity if we say that complex number (non-real) is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

(iii) In real numbers, if $a^2 + b^2 = 0$, then $a = 0 = b$ but in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Illustration 2 : $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if θ =

(A) $2n\pi \pm \dfrac{\pi}{3}$, $n \in I$ (B) $n\pi + \dfrac{\pi}{3}$, $n \in I$ (C) $n\pi \pm \dfrac{\pi}{3}$, $n \in I$ (D) none of these

Solution : $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if the real part vanishes, i.e.,

$$\frac{(3+2i\sin\theta)}{(1-2i\sin\theta)} \times \frac{(1+2i\sin\theta)}{(1+2i\sin\theta)} = \frac{\left(3-4\sin^2\theta\right)+i\left(8\sin\theta\right)}{\left(1+4\sin^2\theta\right)}$$

$$\frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0 \quad \Rightarrow \quad 3 - 4\sin^2\theta = 0 \text{ (only if } \theta \text{ be real)}$$

$$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\sin\frac{\pi}{3}\right)^2$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3},\ n \in I$$

Ans. (C)

> **Do yourself - 1 :**
>
> **(i)** Determine least positive value of n for which $\left(\dfrac{1+i}{1-i}\right)^n = 1$
>
> **(ii)** Find the value of the sum $\sum_{n=1}^{5}(i^n + i^{n+2})$, where $i = \sqrt{-1}$.

3. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts are respectively equal.

Illustration 3 : The values of x and y satisfying the equation $\dfrac{(1+i)x-2i}{3+i} + \dfrac{(2-3i)y+i}{3-i} = i$ are

(A) $x = -1$, $y = 3$ (B) $x = 3$, $y = -1$ (C) $x = 0$, $y = 1$ (D) $x = 1$, $y = 0$

Solution : $\dfrac{(1+i)x-2i}{3+i} + \dfrac{(2-3i)y+i}{3-i} = i \quad \Rightarrow \quad (4 + 2i)x + (9 - 7i)y - 3i - 3 = 10i$

Equating real and imaginary parts, we get $2x - 7y = 13$ and $4x + 9y = 3$.

Hence $x = 3$ and $y = -1$. **Ans.(B)**

Illustration 4 : Find the square root of 7 + 24 i.

Solution : Let $\sqrt{7+24i} = a + ib$

Squaring $a^2 - b^2 + 2iab = 7 + 24i$

Compare real & imaginary parts $a^2 - b^2 = 7$ & $2ab = 24$

By solving these two equations

We get $a = \pm 4$, $b = \pm 3$

$\sqrt{7+24i} = \pm(4 + 3i)$

Illustration 5 : If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution : We have , $x = -5 + 2\sqrt{-4}$

$\Rightarrow \quad x + 5 = 4i \quad \Rightarrow \quad (x + 5)^2 = 16i^2$

$\Rightarrow \quad x^2 + 10x + 25 = -16 \quad \Rightarrow \quad x^2 + 10x + 41 = 0$

Now,

$x^4 + 9x^3 + 35x^2 - x + 4$

$\Rightarrow \quad x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$\Rightarrow \quad x^2(0) - x(0) + 4(0) - 160 \quad \Rightarrow \quad -160$ **Ans.**

Do yourself - 2 :

(i) Find the value of $x^3 + 7x^2 - x + 16$, where $x = 1 + 2i$.

(ii) If $a + ib = \frac{c+i}{c-i}$, where c is a real number, then prove that : $a^2 + b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.

(iii) Find square root of $-15 - 8i$

4. THREE IMPORTANT TERMS : CONJUGATE/MODULUS/ARGUMENT :

(a) CONJUGATE COMPLEX :

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,\text{Re}(z)$

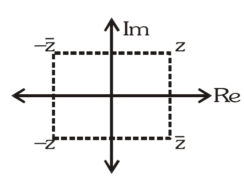
(ii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z\bar{z} = a + b$, which is purely real

(iv) If z is purely real, then $z - \bar{z} = 0$

(v) If z is purely imaginary, then $z + \bar{z} = 0$

(vi) If z lies in the 1st quadrant, then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.



(b) Modulus :

If P denotes complex number $z = x + iy$, then the length OP is called modulus of complex number z. It is denoted by $|z|$.

$OP = |z| = \sqrt{x^2 + y^2}$

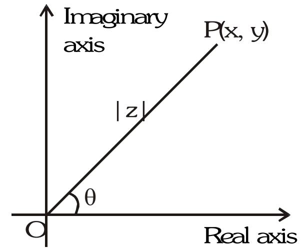
Geometrically $|z|$ represents the distance of point P from origin. ($|z| \geq 0$)

Note : Unlike real numbers, $|z| = \begin{cases} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{cases}$

is not correct.

(c) **Argument or Amplitude :**

If P denotes complex number $z = x + iy$ and if OP makes an angle θ with real axis, then θ is called one of the arguments of z.



$\theta = \tan^{-1}\frac{y}{x}$ (angle made by OP with positive real axis)

Note :

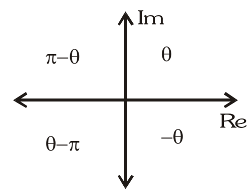
(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number, then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \leq \pi$ is called **Amplitude (principal value of the argument).**

(iii) Principal argument of a complex number $z = x + iy$ can be found out using method given below :

(a) Find $\theta = \tan^{-1}\left|\frac{y}{x}\right|$ such that $\theta \in \left(0, \frac{\pi}{2}\right)$.

(b) Use given figure to find out the principal argument according as the point lies in respective quadrant.



(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) The unique value of $\theta = \tan^{-1}\frac{y}{x}$ such that $0 < \theta \leq 2\pi$ is called **least positive argument.**

(vi) If $z = 0$, arg(z) is not defined

(vii) If z is real & negative, $\arg(z) = \pi$.

(viii) If z is real & positive, $\arg(z) = 0$

(ix) If $\theta = \frac{\pi}{2}$, z lies on the positive side of imaginary axis.

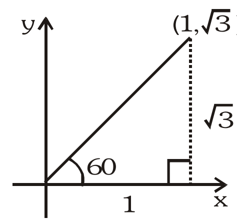
(x) If $\theta = -\frac{\pi}{2}$, z lies on the negative side of imaginary axis.

By specifying the modulus & argument a complex number is defined completely. Argument impart direction & modulus impart distance from origin.

For the complex number $0 + 0i$ the argument is not defined and this is the only complex number which is given by its modulus only.

Illustration 6 : Find the modulus, argument, principal value of argument, least positive argument of complex numbers (a) $1 + i\sqrt{3}$ (b) $-1 + i\sqrt{3}$ (c) $1 - i\sqrt{3}$ (d) $-1 - i\sqrt{3}$

Solution : (a) For $z = 1 + i\sqrt{3}$



$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\arg(z) = 2n\pi + \frac{\pi}{3}$, $n \in I$

Least positive argument is $\frac{\pi}{3}$

If the point is lying in first or second quadrant then amp(z) is taken in anticlockwise direction.

In this case $\text{amp}(z) = \frac{\pi}{3}$

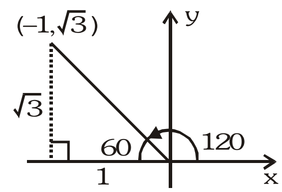
(b) For $z = -1 + i\sqrt{3}$

$|z| = 2$

$\arg(z) = 2n\pi + \frac{2\pi}{3}$, $n \in I$

Least positive argument $= \frac{2\pi}{3}$

$\text{amp}(z) = \frac{2\pi}{3}$

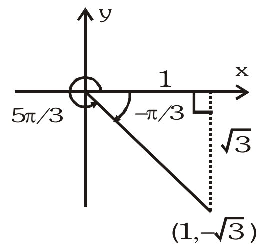


(c) For $z = 1 - i\sqrt{3}$

$|z| = 2$

$\arg(z) = 2n\pi - \frac{\pi}{3}$, $n \in I$

Least positive argument $= \frac{5\pi}{3}$



If the point lies in third or fourth quadrant then consider amp(z) in clockwise direction.

In this case $\text{amp}(z) = -\frac{\pi}{3}$

(d) For $z = -1 - i\sqrt{3}$

$|z| = 2$

$\arg(z) = 2n\pi - \frac{2\pi}{3}$, $n \in I$

Least positive argument $= \frac{4\pi}{3}$

$\text{amp}(z) = -\frac{2\pi}{3}$

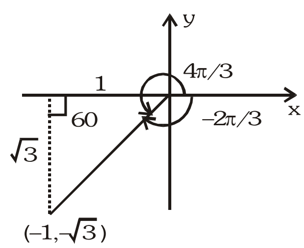


Illustration 7 : Find modulus and argument for $z = 1 - \sin\alpha + i\cos\alpha$, $\alpha \in (0,2\pi)$

Solution : $|z| = \sqrt{(1-\sin\alpha)^2 + (\cos\alpha)^2} = \sqrt{2 - 2\sin\alpha} = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$

Case (i) For $\alpha \in \left(0, \frac{\pi}{2}\right)$, z will lie in I quadrant.

$$\text{amp}(z) = \tan^{-1}\frac{\cos\alpha}{1-\sin\alpha} \Rightarrow \text{amp}(z) = \tan^{-1}\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2} = \tan^{-1}\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$$

$$\Rightarrow \arg z = \tan^{-1}\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

Since $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$\therefore \text{amp}(z) = \left(\frac{\pi}{4} + \frac{\alpha}{2}\right), \ |z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

Case (ii) at $\alpha = \frac{\pi}{2}$: $z = 0 + 0i$

$|z| = 0$

amp (z) is not defined.

Case (iii) For $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, z will lie in IV quadrant

so amp (z) = $-\tan^{-1}\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$

Since $\frac{\alpha}{2}+\frac{\pi}{4} \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore$ amp (z) = $-\left(\frac{\alpha}{2}+\frac{\pi}{4}-\pi\right) = \frac{3\pi}{4}-\frac{\alpha}{2}$, $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\right)$

Case (iv) at $\alpha = \frac{3\pi}{2}$: z = 2 + 0i

$|z| = 2$

amp (z) = 0

Case (v) For $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$

z will lie in I quadrant

arg (z) = $\tan^{-1}\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$

Since $\frac{\alpha}{2}+\frac{\pi}{4} \in \left(\pi, \frac{5\pi}{4}\right)$

$\therefore$ arg z = $\frac{\alpha}{2}+\frac{\pi}{4}-\pi = \frac{\alpha}{2}-\frac{3\pi}{4}$, $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\right)$

Do yourself - 3 :

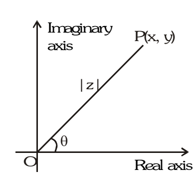
Find the modulus and amplitude of following complex numbers :

(i) $-2+2\sqrt{3}i$ (ii) $-\sqrt{3}-i$ (iii) $-2i$ (iv) $\frac{1+2i}{1-3i}$ (v) $\frac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

5. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometrical Representation) :

Every complex number z = x + i y can be represented by a point on the cartesian plane known as complex plane by the ordered pair (x , y). There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.



For z = x + iy; $|z| = \sqrt{x^2+y^2}$; $\bar{z} = x - iy$ and $\theta = \tan^{-1}\frac{y}{x}$

Note :

(i) Distance between the two complex numbers z_1 & z_2 is given by $|z_1 - z_2|$.

(ii) $|z - z_0| = r$, represents a circle, whose centre is z_0 and radius is r.

Illustration 8 : Find the locus of :

(a) $|z - 1|^2 + |z + 1|^2 = 4$ (b) $Re(z^2) = 0$

Solution : (a) Let z = x + iy

$\Rightarrow$ $(|x + iy - 1|)^2 + (|x + iy + 1|)^2 = 4$

$\Rightarrow$ $(x - 1)^2 + y^2 + (x + 1)^2 + y^2 = 4$

$\Rightarrow$ $x^2 - 2x + 1 + y^2 + x^2 + 2x + 1 + y^2 = 4 \Rightarrow x^2 + y^2 = 1$

Above represents a circle on complex plane with center at origin and radius unity.

(b) Let z = x + iy

$\Rightarrow$ $z^2 = x^2 - y^2 + 2xyi$

$\therefore$ $Re(z^2) = 0$

$\Rightarrow$ $x^2 - y^2 = 0 \Rightarrow y = \pm x$

Thus $Re(z^2) = 0$ represents a pair of straight lines passing through origin.

Illustration 9 : If z is a complex number such that $z^2 = (\overline{z})^2$, then

(A) z is purely real (B) z is purely imaginary

(C) either z is purely real or purely imaginary (D) none of these

Solution : Let z = x + iy, then its conjugate $\overline{z} = x - iy$

Given that $z^2 = (\overline{z})^2 \Rightarrow x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy \Rightarrow 4ixy = 0$

If $x \neq 0$ then y = 0 and if $y \neq 0$ then x = 0. **Ans. (C)**

Illustration 10 : Among the complex number z which satisfies $|z - 25i| \leq 15$, find the complex numbers z having

(a) least positive argument (b) maximum positive argument

(c) least modulus (d) maximum modulus

Solution : The complex numbers z satisfying the condition

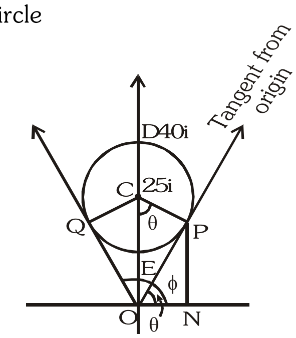
$|z - 25i| \leq 15$

are represented by the points inside and on the circle of radius 15 and centre at the point C(0, 25).

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle

Here θ = least positive argument

and ϕ = maximum positive argument



$\therefore$ In $\Delta OCP, OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$

and $\sin\theta = \frac{OP}{OC} = \frac{20}{25} = \frac{4}{5}$

$\therefore$ $\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\therefore$ $z_p = 20(\cos\theta + i\sin\theta) = 20\left(\frac{3}{5} + i\frac{4}{5}\right)$

$\therefore$ $z_p = 12 + 16i$

Similarly $z_Q = -12 + 16i$

From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10i$

and $z_D = \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = 25i + 15i = 40i$

Do yourself - 4 :

(i) Find the distance between two complex numbers $z_1 = 2 + 3i$ & $z_2 = 7 - 9i$ on the complex plane.

(ii) Find the locus of $|z - 2 - 3i| = 1$.

(iii) If z is a complex number, then $z^2 + \overline{z}^2 = 2$ represents -

(A) a circle (B) a straight line (C) a hyperbola (D) an ellipse

(c) Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as $\text{CiS}\,\theta$.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

It was introduced by Euler in 1748, and is used as a method of expressing complex numbers.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

(d) Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

Illustration 11 : Express the following complex numbers in polar and exponential form :

(i) $\dfrac{1+3i}{1-2i}$ (ii) $\dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

Solution : (i) Let $z = \dfrac{1+3i}{1-2i} = \dfrac{1+3i}{1-2i} \times \dfrac{1+2i}{1+2i} = -1 + i$

$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\tan\alpha = \left|\dfrac{1}{-1}\right| = 1 = \tan\dfrac{\pi}{4} \Rightarrow \alpha = \dfrac{\pi}{4}$

$\because$ Re(z) < 0 and Im(z) > 0 $\Rightarrow$ z lies in second quadrant.

$\therefore$ $\theta = \arg(z) = \pi - \alpha = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

Hence Polar form is $z = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

and exponential form is $z = \sqrt{2}\,e^{3\pi/4}$

(ii) Let $z = \dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \dfrac{i-1}{\frac{1}{2} + \frac{i\sqrt{3}}{2}} = \dfrac{2(i-1)}{(1+i\sqrt{3})}$

$\Rightarrow$ $z = \dfrac{2(i-1)}{(1+i\sqrt{3})} \times \dfrac{(1-i\sqrt{3})}{(1-i\sqrt{3})}$ $\Rightarrow z = \left(\dfrac{\sqrt{3}-1}{2}\right) + i\left(\dfrac{\sqrt{3}+1}{2}\right)$

$\because$ Re(z) > 0 and Im(z) > 0 $\Rightarrow$ z lies in first quadrant.

$\therefore$ $|z| = \sqrt{\left(\dfrac{\sqrt{3}-1}{2}\right)^2 + \left(\dfrac{\sqrt{3}+1}{2}\right)^2} = \sqrt{\dfrac{2(3+1)}{4}} = \sqrt{2}$.

$\tan\theta = \left|\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right| = \tan\dfrac{5\pi}{12} \Rightarrow \alpha = \dfrac{5\pi}{12}$

Hence Polar form is $z = \sqrt{2}\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$

and exponential form is $z = \sqrt{2}e^{5\pi/12}$

Illustration 12 : If $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right)$ then $x_1x_2x_3$.........∞ is equal to -

(A) –1 (B) 1 (C) 0 (D) ∞

Solution : $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right) = 1 \quad e^{i\frac{\pi}{2^n}}$

$x_1x_2x_3.........\infty$

$= e^{i\frac{\pi}{2^1}}.e^{i\frac{\pi}{2^2}} --- e^{i\frac{\pi}{2^n}} = e^{i\left(\frac{\pi}{2}+\frac{\pi}{2^2}+---+\frac{\pi}{2^n}\right)}$

$= \cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+........\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+........\right) = -1$

$\left(\text{as } \frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+........ = \frac{\pi/2}{1-1/2} = \pi\right)$ **Ans. (A)**

Do yourself - 5 :

Express the following complex number in polar form and exponential form :

(i) $-2 + 2i$ (ii) $-1-\sqrt{3}i$ (iii) $\frac{(1+7i)}{(2-i)^2}$ (iv) $(1 - \cos\theta + i\sin\theta)$, $\theta \in (0,\pi)$

6. IMPORTANT PROPERTIES OF CONJUGATE :

(a) $z + \bar{z} = 2\,\text{Re}(z)$ (b) $z - \bar{z} = 2i\,\text{Im}(z)$ (c) $\overline{(\bar{z})} = z$

(d) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (e) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(f) $\overline{z_1\, z_2} = \bar{z}_1 \,.\, \bar{z}_2$. In general $\overline{z_1z_2.........z_n} = \bar{z}_1.\bar{z}_2.........\bar{z}_n$

(g) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$ (h) If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

7. IMPORTANT PROPERTIES OF MODULUS :

(a) $|z| \geq 0$ (b) $|z| \geq \text{Re}(z)$ (c) $|z| \geq \text{Im}(z)$

(d) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$ (e) $z\bar{z} = |z|^2$

(f) $|z_1\, z_2| = |z_1|.|z_2|$. In general $|z_1z_2.........z_n| = |z_1|.|z_2|.........|z_n|$

(g) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(h) $|z^n| = |z|^n$, $n \in I$

(i) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

(j) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\alpha - \beta)$, where α, β are $\arg(z_1)$, $\arg(z_2)$ respectively.

(k) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

(l) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ [Triangle Inequality]

(m) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ [Triangle Inequality]

8. IMPORTANT PROPERTIES OF AMPLITUDE :

(a) amp $(z_1 . z_2)$ = amp z_1 + amp z_2 + $2k\pi$; $k \in I$

(b) amp $\left(\frac{z_1}{z_2}\right)$ = amp z_1 − amp z_2 + $2k\pi$; $k \in I$

(c) amp(z^n) = n amp(z) + $2k\pi$; $n,k \in I$

where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

Illustration 13 : Find amp z and |z| if $z = \left[\frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right]^2$.

Solution : amp z = $2\left[\text{amp}(3+4i)+\text{amp}(1+i)+\text{amp}(1+\sqrt{3}i)-\text{amp}(1-i)-\text{amp}(4-3i)-\text{amp}(2i)\right]+2k\pi$

where $k \in I$ and k chosen so that amp z lies in $(-\pi,\pi]$.

$$\Rightarrow \quad \text{amp}\, z = 2\left[\tan^{-1}\frac{4}{3}+\frac{\pi}{4}+\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)-\tan^{-1}\left(-\frac{3}{4}\right)-\frac{\pi}{2}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp}\, z = 2\left[\tan^{-1}\frac{4}{3}+\cot^{-1}\frac{4}{3}+\frac{\pi}{3}\right]+2k\pi \Rightarrow \text{amp}\, z = 2\left[\frac{\pi}{2}+\frac{\pi}{3}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp}\, z = -\frac{\pi}{3} \quad [\text{at } k = -1]$$ **Ans.**

Also,

$$|z| = \left|\frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right|^2 \quad \Rightarrow \quad |z| = \left(\frac{|3+4i||1+i||1+\sqrt{3}i|}{|1-i|\,|4-3i|\,|2i|}\right)^2$$

$$\Rightarrow \quad |z| = \left(\frac{5\times\sqrt{2}\times 2}{\sqrt{2}\times 5\times 2}\right)^2 = 1$$ **Ans.**

Aliter

$$z = \left[\frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right]^2 \quad \Rightarrow \quad z = \left[-\frac{\sqrt{3}+i}{2}\right]^2 \Rightarrow z = \frac{2-2\sqrt{3}i}{4} = \frac{1}{2}-\frac{\sqrt{3}i}{2}$$

Hence |z| = 1, amp(z) $= -\frac{\pi}{3}$.

Illustration 14 : If $\left|\frac{z-i}{z+i}\right| = 1$, then locus of z is -

(A) x-axis (B) y-axis (C) x = 1 (D) y = 1

Solution : We have, $\left|\frac{z-i}{z+i}\right| = 1 \Rightarrow \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = 1$

$$\Rightarrow \frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = 1 \Rightarrow x^2+(y-1)^2 = x^2+(y+1)^2 \Rightarrow 4y = 0; y = 0$$, which is x-axis **Ans. (A)**

Illustration 15 : If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then $\left(\frac{z_1}{z_2}\right)$ is -

(A) zero or purely imaginary (B) purely imaginary
(C) purely real (D) none of these

Solution : Here let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), |z_1| = r_1$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2), |z_2| = r_2$

$\therefore \quad |(z_1 + z_2)|^2 = \left|(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)\right|^2$

$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = |z_1|^2 + |z_2|^2$ if $\cos(\theta_1 - \theta_2) = 0$

$\therefore \quad \theta_1 - \theta_2 = \pm\frac{\pi}{2}$

$\Rightarrow \quad \text{amp}(z_1) - \text{amp}(z_2) = \pm\frac{\pi}{2} \quad \Rightarrow \quad \text{amp}\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \frac{z_1}{z_2}$ is purely imaginary **Ans. (B)**

Illustration 16 : z_1 and z_2 are two complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}$ is unimodular (whose modulus is one), while z_2 is not unimodular. Find $|z_1|$.

Solution : Here $\left|\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}\right| = 1 \Rightarrow \frac{|z_1 - 2z_2|}{|2 - z_1\overline{z_2}|} = 1$

$\Rightarrow \quad |z_1 - 2z_2| = |2 - z_1\overline{z_2}| \Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\overline{z_2}|^2$

$\Rightarrow \quad (z_1 - 2z_2)(\overline{z_1 - 2z_2}) = (2 - z_1\overline{z_2})(\overline{2 - z_1\overline{z_2}})$

$\Rightarrow \quad (z_1 - 2z_2)(\overline{z_1} - 2\overline{z_2}) = (2 - z_1\overline{z_2})(2 - \overline{z_1}z_2)$

$\Rightarrow \quad z_1\overline{z_1} - 2z_1\overline{z_2} - 2z_2\overline{z_1} + 4z_2\overline{z_2} = 4 - 2\overline{z_1}z_2 - 2z_1\overline{z_2} + z_1\overline{z_1}z_2\overline{z_2}$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2 \Rightarrow |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$

$\Rightarrow \quad (|z_1|^2 - 4)(1 - |z_2|^2) = 0$

But $|z_2| \neq 1$ (given)

$\therefore \quad |z_1|^2 = 4$

Hence, $|z_1| = 2$.

Illustration 17 : The locus of the complex number z in argand plane satisfying the inequality

$\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 \quad \left(\text{where } |z-1| \neq \frac{2}{3}\right)$ is -

(A) a circle (B) interior of a circle (C) exterior of a circle (D) none of these

Solution : We have, $\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 = \log_{1/2}\left(\frac{1}{2}\right)$

$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2} \qquad [\because \log_a x$ is a decreasing function if $a < 1]$

$\Rightarrow 2|z-1| + 8 < 3|z-1| - 2$ as $|z - 1| > 2/3$

$\Rightarrow |z-1| > 10$

which is exterior of a circle. **Ans. (C)**

Illustration 18 : If $\left|z - \frac{4}{z}\right| = 2$, then the greatest value of $|z|$ is -

(A) $1 + \sqrt{2}$ (B) $2 + \sqrt{2}$ (C) $\sqrt{3} + 1$ (D) $\sqrt{5} + 1$

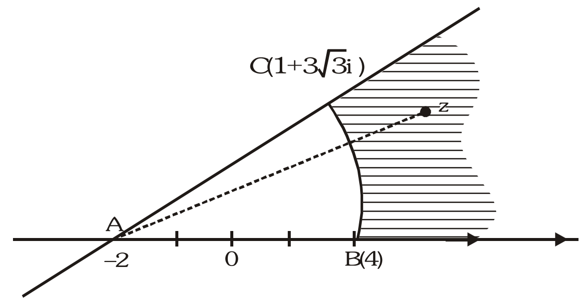
Solution : We have $|z| = \left|z - \frac{4}{z} + \frac{4}{z}\right| \le \left|z - \frac{4}{z}\right| + \frac{4}{|z|} = 2 + \frac{4}{|z|}$

$\Rightarrow \quad |z|^2 \le 2|z| + 4 \Rightarrow \quad (|z| - 1)^2 \le 5$

$\Rightarrow \quad |z| - 1 \le \sqrt{5} \quad \Rightarrow \quad |z| \le \sqrt{5} + 1$

Therefore, the greatest value of $|z|$ is $\sqrt{5} + 1$. **Ans. (D)**

Illustration 19 : Shaded region is given by -



(A) $|z + 2| \ge 6,\ 0 \le \arg(z) \le \frac{\pi}{6}$

(B) $|z + 2| \ge 6,\ 0 \le \arg(z) \le \frac{\pi}{3}$

(C) $|z + 2| \le 6,\ 0 \le \arg(z) \le \frac{\pi}{2}$

(D) None of these

Solution : Note that AB = 6 and $1 + 3\sqrt{3}i = -2 + 3 + 3\sqrt{3}i = -2 + 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2 + 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\therefore \quad \angle BAC = \frac{\pi}{3}$

Thus, shaded region is given by $|z + 2| \ge 6$ and $0 \le \arg(z + 2) \le \frac{\pi}{3}$ **Ans. (C)**

Do yourself - 6 :

(i) The inequality $|z - 4| < |z - 2|$ represents region given by -

(A) Re(z) > 0 (B) Re(z) < 0 (C) Re(z) > 3 (D) none

(ii) If $z = re^{i\theta}$, then the value of $|e^{iz}|$ is equal to -

(A) $e^{-r\cos\theta}$ (B) $e^{r\cos\theta}$ (C) $e^{r\sin\theta}$ (D) $e^{-r\sin\theta}$

9. SECTION FORMULA AND COORDINATES OF ORTHOCENTRE, CENTROID, CIRCUMCENTRE, INCENTRE OF A TRIANGLE :

If z_1 & z_2 are two complex numbers then the complex number $z = \frac{nz_1 + mz_2}{m + n}$ divides the join of z_1 & z_2 in the ratio m : n.

Note :

(i) If a , b , c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

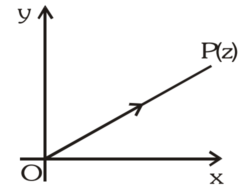
(ii) If the vertices A, B, C of a triangle represent the complex numbers z_1, z_2, z_3 respectively, then :

- Centroid of the Δ ABC = $\frac{z_1 + z_2 + z_3}{3}$
- Orthocentre of the Δ ABC =

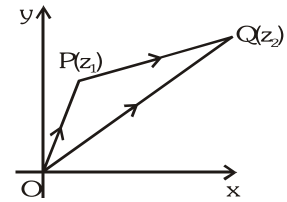
$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \quad \text{or} \quad \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

- Incentre of the ΔABC = $\frac{(az_1 + bz_2 + cz_3)}{(a + b + c)}$
- Circumcentre of the ΔABC = $\frac{(z_1\sin 2A + z_2\sin 2B + z_3\sin 2C)}{(\sin 2A + \sin 2B + \sin 2C)}$

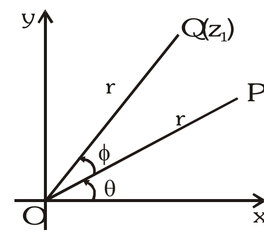
10. VECTORIAL REPRESENTATION OF A COMPLEX NUMBER :



(a) In complex number every point can be represented in terms of position vector. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

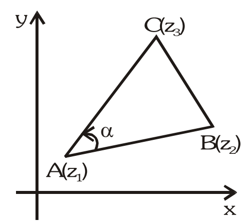


(b) If $P(z_1)$ & $Q(z_2)$ be two complex numbers on argand plane then $\vec{PQ}$ represents complex number $z_2 - z_1$.



Note :

(i) If $\vec{OP} = z = re^{i\theta}$ then $\vec{OQ} = z_1 = re^{i(\theta+\phi)} = z.\, e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\, e^{i\phi}$ i.e. $\frac{z_1}{|z_1|} = \frac{z}{|z|}e^{i\phi}$

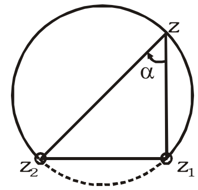


(ii) In general, if z_1, z_2, z_3 be the three vertices of ΔABC then $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$. Here $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \alpha$.

(iii) Note that the locus of z satisfying $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ is:

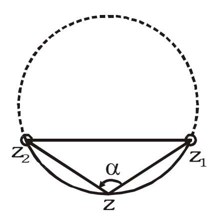
Case (a) $0 < \alpha < \pi/2$

Locus is major arc of circle as shown excluding z_1 & z_2



Case (b) $\frac{\pi}{2} < \alpha < \pi$

Locus is minor arc of circle as shown excluding z_1 & z_2



(iv) If A, B, C & D are four points representing the complex numbers z_1, z_2, z_3 & z_4 then AB || CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

AB ⊥ CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary.

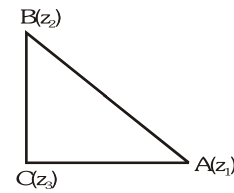
(v) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

(1) $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$ (2) $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

Illustration 20 : Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Solution : In the isosceles triangle ABC, AC = BC and BC⊥AC. It means that AC is rotated through angle $\pi/2$ to occupy the position BC.

Hence we have, $\frac{z_2 - z_3}{z_1 - z_3} = e^{+i\pi/2} = +i \Rightarrow z_2 - z_3 = +i(z_1 - z_3)$



$\Rightarrow \quad z_2^2 + z_3^2 - 2z_2z_3 = -\left(z_1^2 + z_3^2 - 2z_1z_3\right)$

$\Rightarrow \quad z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2$

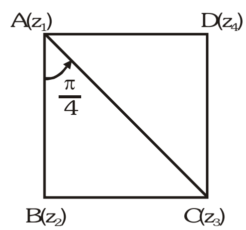
$= 2\left(z_1 - z_3\right)\left(z_3 - z_2\right)$

$\Rightarrow \quad \left(z_1 - z_2\right)^2 = 2\left(z_1 - z_3\right)\left(z_3 - z_2\right)$

Illustration 21: If the vertices of a square ABCD are z_1, z_2, z_3 & z_4 then find z_3 & z_4 in terms of z_1 & z_2.

Solution : Using vector rotation at angle A

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\frac{\pi}{4}}$$



$\because \ |z_3 - z_1| = AC$ and $|z_2 - z_1| = AB$

Also AC = $\sqrt{2}$ AB

$\therefore \ |z_3 - z_1| = \sqrt{2}\,|z_2 - z_1|$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

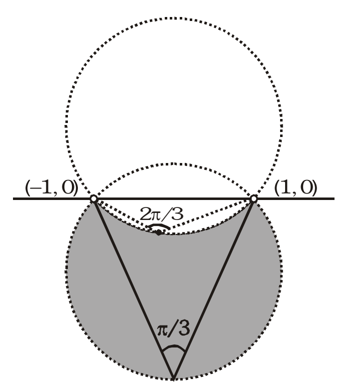
$\Rightarrow z_3 - z_1 = (z_2 - z_1)(1 + i)$

$\Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$

Similarly $z_4 = z_2 + (1 + i)(z_1 - z_2)$

Illustration 22 : Plot the region represented by $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ in the Argand plane.

Solution : Let us take $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$, clearly z lies on the minor arc of the circle passing through (1, 0) and (–1, 0). Similarly, $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{3}$ means that 'z' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now if we take any point in the region included between two arcs say $P_1(z_1)$ we get



$$\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$$

Thus $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ represents the shaded region (excluding points (1, 0) and (–1, 0)) .

Do yourself - 7 :

(i) A complex number $z = 3 + 4i$ is rotated about another fixed complex number $z_1 = 1 + 2i$ in anticlockwise direction by 45 angle. Find the complex number represented by new position of z in argand plane.

(ii) If A, B, C are three points in argand plane representing the complex number z_1, z_2, z_3 such that $z_1 = \dfrac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in R$, then find the distance of point A from the line joining points B and C.

(iii) If $A(z_1)$, $B(z_2)$, $C(z_3)$ are vertices of ΔABC in which $\angle ABC = \dfrac{\pi}{4}$ and $\dfrac{AB}{BC} = \sqrt{2}$, then find z_2 in terms of z_1 and z_3.

(iv) If a & b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle then a and b are equal to :-

(A) $a = b = 1/2$ (B) $a = b = 2 - \sqrt{3}$ (C) $a = b = -2 + \sqrt{3}$ (D) $a = b = \sqrt{2} - 1$

(v) If $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$, find locus of z.

11. DE'MOIVRE'S THEOREM :

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of the roots of a complex quantity should be determined by using theory of equations.

Illustration 23: If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and also $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

(a) $\cos2\alpha + \cos2\beta + \cos2\gamma = \sin2\alpha + \sin2\beta + \sin2\gamma = 0$

(b) $\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

(c) $\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha + \beta + \gamma)$

Solution : Let $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$ & $z_3 = \cos\gamma + i\sin\gamma$.

$\therefore$ $z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i \cdot 0 = 0$ .... (i)

(a) Also $\dfrac{1}{z_1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$

$$\frac{1}{z_2} = \cos\beta - i\sin\beta,\ \frac{1}{z_3} = \cos\gamma - i\sin\gamma$$

$\therefore$ $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$ .... (ii)

$= 0 - i \cdot 0 = 0$

Now $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

$= 0 - 2z_1z_2z_3\left(\dfrac{1}{z_3} + \dfrac{1}{z_1} + \dfrac{1}{z_2}\right) = 0 - 2z_1z_2z_3 \cdot 0 = 0$ {using (i) and (ii)}

or $(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

or $\cos2\alpha + i\sin2\alpha + \cos2\beta + i\sin2\beta + \cos2\gamma + i\sin2\gamma = 0 + i.0$

Equating real and imaginary parts on both sides,

$\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ and $\sin2\alpha + \sin2\beta + \sin2\gamma = 0$

(b) If $z_1 + z_2 + z_3 = 0$ then $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

$\therefore$ $(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

or $\cos3\alpha + i\sin3\alpha + \cos3\beta + i\sin3\beta + \cos3\gamma + i\sin3\gamma$

$= 3\{\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\}$

Equating imaginary parts on both sides, $\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha+\beta+\gamma)$

(c) Equating real parts on both sides, $\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha+\beta+\gamma)$

Do yourself - 8 :

(i) If $z_r = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}$, $r = 0,1,3,4,\ldots\ldots$, then $z_1z_2z_3z_4z_5$ is equal to -

(A) –1 (B) 0 (C) 1 (D) none of these

(ii) If $(x-1)^4 - 16 = 0$, then the sum of nonreal complex values of x is -

(A) 2 (B) 0 (C) 4 (D) none of these

(iii) If $(\sqrt{3} - i)^n = 2^n$, $n \in Z$, then n is a multiple of -

(A) 6 (B) 10 (C) 9 (D) 12

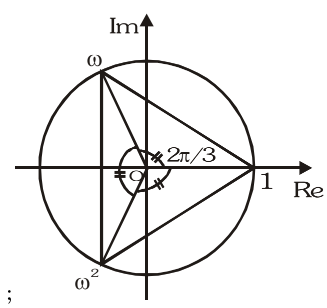
12. CUBE ROOT OF UNITY :

(a) The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}(\omega)$, $\frac{-1-i\sqrt{3}}{2}(\omega^2)$.

(b) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3 & $1 + \omega^r + \omega^{2r} = 3$ if $r = 3\lambda$; $\lambda \in I$

(c) In polar form the cube roots of unity are :

$1 = \cos 0 + i\sin 0$; $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$



(d) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

(e) The following factorisation should be remembered :

(a, b, c $\in$ R & ω is the cube root of unity)

$a^3 - b^3 = (a-b)(a-\omega b)(a-\omega b)$; $x^2 + x + 1 = (x-\omega)(x-\omega^2)$;

$a^3 + b^3 = (a+b)(a+\omega b)(a+\omega^2 b)$;

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+\omega b+\omega c)(a+\omega b+\omega c)$

Illustration 24 : If α & β are imaginary cube roots of unity then $\alpha^n + \beta^n$ is equal to -

(A) $2\cos\frac{2n\pi}{3}$ (B) $\cos\frac{2n\pi}{3}$ (C) $2i\sin\frac{2n\pi}{3}$ (D) $i\sin\frac{2n\pi}{3}$

Solution :

$$\alpha = \frac{\cos2\pi}{3} + \frac{i\sin2\pi}{3}$$

$$\beta = \frac{\cos2\pi}{3} - \frac{i\sin2\pi}{3}$$

$$\alpha^n + \beta^n = \left(\frac{\cos2\pi}{3} + \frac{i\sin2\pi}{3}\right)^n + \left(\frac{\cos2\pi}{3} - \frac{i\sin2\pi}{3}\right)^n$$

$$= \left(\frac{\cos2n\pi}{3} + \frac{i\sin2n\pi}{3}\right) + \left(\frac{\cos2n\pi}{3} - i\sin\left(\frac{2n\pi}{3}\right)\right) = 2\cos\left(\frac{2n\pi}{3}\right)$$

Ans. (A)

Illustration 25 : If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is imaginary cube root of unity), then find the value of $\frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1}$.

Solution : We have $x^3 - 3x^2 + 3x + 7 = 0$

$\therefore$ $(x-1)^3 + 8 = 0$

$\therefore$ $(x-1)^3 = (-2)^3$

$$\Rightarrow \left(\frac{x-1}{-2}\right)^3 = 1 \Rightarrow \frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2 \text{ (cube roots of unity)}$$

$\therefore$ $x = -1, 1-2\omega, 1-2\omega^2$

Here $\alpha = -1$, $\beta = 1-2\omega$, $\gamma = 1-2\omega^2$

$\therefore$ $\alpha - 1 = -2$, $\beta - 1 = -2\omega$, $\gamma - 1 = -2\omega^2$

$$\text{Then } \frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right)+\left(\frac{-2\omega}{-2\omega^2}\right)+\left(\frac{-2\omega^2}{-2}\right) = \frac{1}{\omega}+\frac{1}{\omega}+\omega^2 = \omega^2+\omega^2+\omega^2$$

$$\text{Therefore } \frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1} = 3\omega^2.$$ **Ans.**

Do yourself - 9 :

(i) If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^2$ equals : -

(A) ω (B) -4ω (C) ω^2 (D) 4ω

(ii) If ω is a non real cube root of unity, then the expression $(1-\omega)(1-\omega^2)(1+\omega^4)(1+\omega^8)$ is equal to : -

(A) 0 (B) 3 (C) 1 (D) 2

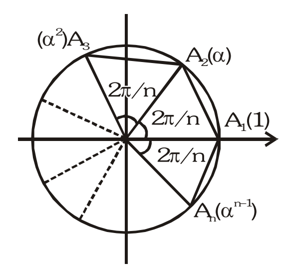
13. n^{th} ROOTS OF UNITY :

If $1, \alpha_1, \alpha_2, \alpha_3 \ldots \alpha_{n-1}$ are the n, n^{th} root of unity then :

(a) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(b) Their arguments are in A.P. with common difference $\frac{2\pi}{n}$

(c) The points represented by n, n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having center at origin, one vertex being on positive real axis.



(d) $1^p + \alpha_1^p + \alpha_2^p + \ldots + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n
$= n$ if p is an integral multiple of n

(e) $(1-\alpha_1)(1-\alpha_2)\ldots\ldots(1-\alpha_{n-1}) = n$

(f) $(1+\alpha_1)(1+\alpha_2)\ldots\ldots(1+\alpha_{n-1}) = 0$ if n is even and
$= 1$ if n is odd.

(g) $1.\alpha_1.\alpha_2.\alpha_3\ldots\ldots\alpha_{n-1} = 1$ or -1 according as n is odd or even.

Illustration 26: Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - \cos\frac{2\pi k}{7}\right)$

Solution :

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right) - \sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right) = \sum_{k=1}^{6}\sin\frac{2\pi k}{7} - \sum_{k=0}^{6}\cos\frac{2\pi k}{7} + 1$$

$= \sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$-\sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) $+1 = 0 - 0 + 1 = 1$

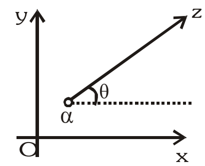
14. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

(a) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\dfrac{n+1}{2}\right)\theta.$

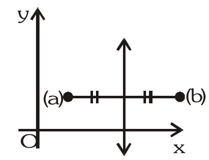
(b) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\dfrac{n+1}{2}\right)\theta.$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

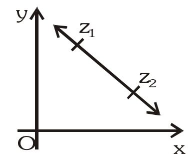
15. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :



(a) $\text{amp}(z-\alpha) = \theta$ is a ray emanating from the complex point α and inclined at an angle θ to the x– axis.

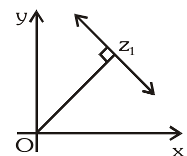


(b) $|z-a| = |z-b|$ is the perpendicular bisector of the segment joining a & b.



(c) The equation of a line joining z_1 & z_2 is given by ;

$z = z_1 + t(z_1 - z_2)$ where t is a parameter.

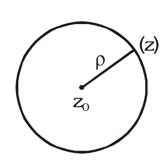


(d) $z = z_1(1 + it)$ where t is a real parameter, is a line through the point z_1 & perpendicular to z_1.

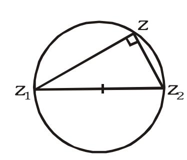
(e) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers to be collinear.

(f) Complex equation of a straight line through two given points z_1 & z_2 can be written as $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$, which on manipulating takes the form as $\bar{\alpha} z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.



(g) The equation of circle having centre z_0 & radius ρ is :
$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho = 0$ which is of the form
$z\bar{z} + \bar{\alpha} z + \alpha\bar{z} + r = 0$, r is real centre $= -\alpha$ & radius $= \sqrt{\alpha\bar{\alpha} - r}$.
Circle will be real if $\alpha\bar{\alpha} - r \ge 0$.



(h) $\arg\left(\dfrac{z - z_2}{z - z_1}\right) = \pm\dfrac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$
this equation represents the circle described on the line segment joining z_1 & z_2 as diameter.

(i) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is, the number $\dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z_4 - z_2}{z_4 - z_1}$ is real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real $\Rightarrow \dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \dfrac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$

Miscellaneous Illustration :

Illustration 27: If z is a point on the Argand plane such that $|z - 1| = 1$, then $\frac{z-2}{z}$ is equal to -

(A) tan (arg z) (B) cot (arg z) (C) i tan (arg z) (D) none of these

Solution : Since $|z - 1| = 1$,

$\therefore$ let $z - 1 = \cos\theta + i\sin\theta$

Then, $z - 2 = \cos\theta + i\sin\theta - 1$

$= -2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ .... (i)

and $z = 1 + \cos\theta + i\sin\theta$

$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ .... (ii)

From (i) and (ii), we get $\frac{z-2}{z} = i\tan\frac{\theta}{2} = i\tan(\arg z)$ $\left(\because \arg z = \frac{\theta}{2} \text{ from (ii)}\right)$ **Ans. (C)**

Illustration 28: Let a be a complex number such that $|a| < 1$ and $z_1, z_2, \ldots\ldots, z_n$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + \ldots a^k$, then show that vertices of the polygon lie within the circle $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$.

Solution : We have, $z_k = 1 + a + a^2 + \ldots.. + a^k = \frac{1-a^{k+1}}{1-a}$

$\Rightarrow z_k - \frac{1}{1-a} = \frac{-a^{k+1}}{1-a} \Rightarrow \left|z_k - \frac{1}{1-a}\right| = \frac{|a|^{k+1}}{|1-a|} < \frac{1}{|1-a|}$ $(\because |a| < 1)$

$\therefore$ Vertices of the polygon $z_1, z_2, \ldots\ldots, z_n$ lie within the circle $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$

Illustration 29 : If z_1 and z_2 are two complex numbers and $C > 0$, then prove that

$$|z_1 + z_2|^2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$$

Solution : We have to prove that : $|z_1 + z_2|^2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$

i.e. $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$

or $z_1\bar{z}_2 + \bar{z}_1 z_2 \le C|z_1|^2 + C^{-1}|z_2|^2$

or $C|z_1|^2 + \frac{1}{C}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \ge 0$ (using Re $(z_1\bar{z}_2) \le |z_1\bar{z}_2|$)

or $\left(\sqrt{C}|z_1| - \frac{1}{\sqrt{C}}|z_2|\right)^2 \ge 0$ which is always true.

Illustration 30 : If $\theta \in [\pi/6, \pi/3]$, $i = 1, 2, 3, 4, 5$ and $z^4\cos\theta_1 + z^3\cos\theta_2 + z^2\cos\theta_3 + z\cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$, then show that $|z| > \frac{3}{4}$

Solution : Given that $\cos\theta_1.z^4 + \cos\theta_2.z^3 + \cos\theta_3.z^2 + \cos\theta_4.z + \cos\theta_5 = 2\sqrt{3}$

or $|\cos\theta_1.z^4 + \cos\theta_2.z^3 + \cos\theta_3.z^2 + \cos\theta_4.z + \cos\theta_5| = 2\sqrt{3}$

$2\sqrt{3} \le |\cos\theta_1.z^4| + |\cos\theta_2.z^3| + |\cos\theta_3.z^2| + |\cos\theta_4.z| + |\cos\theta_5|$

$\because$ $\theta_i \in [\pi/6, \pi/3]$

$\therefore$ $\frac{1}{2} \le \cos\theta_i \le \frac{\sqrt{3}}{2}$

$$2\sqrt{3} \le \frac{\sqrt{3}}{2}|z|^4 + \frac{\sqrt{3}}{2}|z|^3 + \frac{\sqrt{3}}{2}|z|^2 + \frac{\sqrt{3}}{2}|z| + \frac{\sqrt{3}}{2}$$

$$\Rightarrow\ 3 \le |z|^4 + |z|^3 + |z|^2 + |z|$$

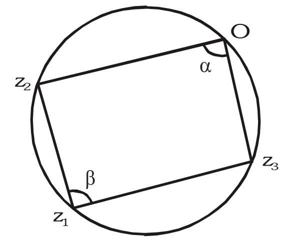
$$\Rightarrow\ 3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + \ldots\ldots\ldots\infty$$

$$\Rightarrow 3 < \frac{|z|}{1-|z|} \Rightarrow 3 - 3|z| < |z|$$

$$\Rightarrow 4|z| > 3 \qquad\qquad \therefore\ \ |z| > \frac{3}{4}$$

Illustration 31 : If z_1, z_2, z_3 are complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

Solution : We have, $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} \Rightarrow \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1} \Rightarrow \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3}$

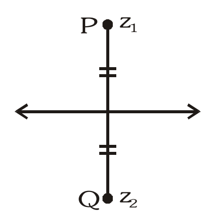


$$\Rightarrow \frac{z_2 - z_1}{z_3 - z_1} = \frac{-z_2}{z_3} \quad \Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(\frac{-z_2}{z_3}\right)$$

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right) \Rightarrow \text{ or } \beta = \pi - \arg\frac{z_3}{z_2} = \pi - \alpha\ = \alpha + \beta = \pi$$

Thus the sum of a pair of opposite angle of a quadrilateral is 180 . Hence, the points 0, z_1, z_2 and z_3 are the vertices of a cyclic quadrilateral i.e. lie on a circle.

Illustration 32 : Two given points P & Q are the reflection points w.r.t. a given straight line if the given line is the right bisector of the segment PQ. Prove that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha} z + \alpha \bar{z} + r = 0$ if and only if ; $\bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.



Solution : Let $P(z_1)$ is the reflection point of $Q(z_2)$ then the perpendicular bisector of z_1 & z_2 must be the line

$$\bar{\alpha} z + \alpha \bar{z} + r = 0 \qquad \ldots\ldots\ldots \text{(i)}$$

Now perpendicular bisector of z_1 & z_2 is, $|z - z_1| = |z - z_2|$

or $(z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$-z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 = -z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2$ ($z\bar{z}$ cancels on either side)

or $(\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} + z_1\bar{z}_1 - z_2\bar{z}_2 = 0$ ……… (ii)

Comparing (i) & (ii) $\frac{\bar{\alpha}}{\bar{z}_2 - \bar{z}_1} = \frac{\alpha}{z_2 - z_1} = \frac{r}{z_1\bar{z}_1 - z_2\bar{z}_2} = \lambda$

$\therefore$ $\bar{\alpha} = \lambda(\bar{z}_2 - \bar{z}_1)$ …….. (iii) $\alpha = \lambda(z_2 - z_1)$ ……… (iv)

$r = \lambda(z_1\bar{z}_1 - z_2\bar{z}_2)$ ……… (v)

Multiplying (iii) by z_1; (iv) by $\bar{z}_2$ and adding

$$\bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$$

Note that we could also multiply (iii) by z_2 & (iv) by $\bar{z}_1$ & add to get the same result.

Hence $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$

Again, let $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$ is true w.r.t. the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$.

Subtracting $\bar{\alpha}\left(z - z_1\right) + \alpha(\bar{z} - \bar{z}_2) = 0$

or $\left|\left(z - z_1\right)\right|\left|\bar{\alpha}\right| = \left|\alpha\right|\left|\left(\bar{z} - \bar{z}_2\right)\right|$ or $\left|z - z_1\right| = \left|\overline{z - z_2}\right| = \left|z - z_2\right|$

Hence 'z' lies on the perpendicular bisector of joins of z_1 & z_2.

ANSWERS FOR DO YOURSELF

1 : (i) $n = 4$ (ii) 0

2 : (i) $-17 + 24i$ (iii) $\pm(1 - 4i)$

3 : (i) $|z| = 4$; $\text{amp}(z) = \frac{2\pi}{3}$ (ii) $|z| = 2; \text{amp}(z) = -\frac{5\pi}{6}$ (iii) $|z| = 2; \text{amp}(z) = -\frac{\pi}{2}$

(iv) $|z| = \frac{1}{\sqrt{2}}; \text{amp}(z) = \frac{3\pi}{4}$ (v) $|z| = 2; \text{amp}(z) = \frac{\pi}{3}$

4 : (i) 13 units (ii) locus is a circle on complex plane with center at (2,3) and radius 1 unit. (iii) C

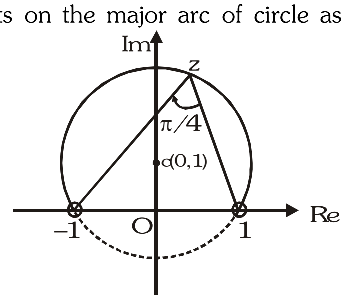
5 : (i) $2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right); 2\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ (ii) $2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right); 2e^{i\left(\frac{4\pi}{3}\right)}$

(iii) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right); \sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ (iv) $2\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right); 2\sin\left(\frac{\theta}{2}\right)e^{i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}$

6 : (i) C (ii) D

7 : (i) $1 + (2 + 2\sqrt{2})i$ (ii) 0 (iii) $z_2 = z_3 + i(z_1 - z_3)$ (iv) B

(v) Locus is all the points on the major arc of circle as shown excluding points 1 & –1.



8 : (i) C (ii) A (iii) D

9 : (i) D (ii) B

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. The value of the sum $\sum_{n=1}^{13}\left(i^n + i^{n+1}\right)$, where $i = \sqrt{-1}$, equals [JEE 98]

(A) i (B) i – 1 (C) –i (D) 0

2. The sequence $S = i + 2i^2 + 3i^3 + \ldots\ldots$ upto 100 terms simplifies to where $i = \sqrt{-1}$ -

(A) 50(1 – i) (B) 25i (C) 25(1 + i) (D) 100(1 – i)

3. Let $i = \sqrt{-1}$. The product of the real part of the roots of $z^2 - z = 5 - 5i$ is -

(A) –25 (B) –6 (C) –5 (D) 25

4. If $z_1 = \frac{1}{a+i}$, $a \neq 0$ and $z_2 = \frac{1}{1+bi}$, $b \neq 0$ are such that $z_1 = \bar{z}_2$ then -

(A) a = 1, b = 1 (B) a = 1, b = –1 (C) a = –1, b = 1 (D) a = –1, b = –1

5. The inequality $|z - 4| < |z - 2|$ represents the following region -

(A) Re(z) > 0 (B) Re(z) < 0 (C) Re(z) > 2 (D) none of these

6. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = \alpha + i\beta$ then $2 . 5 . 10 \ldots (1 + n^2) =$

(A) $\alpha - i\beta$ (B) $\alpha^2 - \beta^2$ (C) $\alpha^2 + \beta^2$ (D) none of these

7. In the quadratic equation $x^2 + (p + iq)x + 3i = 0$, p & q are real. If the sum of the squares of the roots is 8 then :

(A) p = 3, q = –1 (B) p = –3, q = –1

(C) p = 3, q = 1 or p = –3, q = –1 (D) p = –3, q = 1

8. The curve represented by $Re(z^2) = 4$ is -

(A) a parabola (B) an ellipse

(C) a circle (D) a rectangular hyperbola

9. Real part of $e^{e^{i\theta}}$ is -

(A) $e^{\cos\theta}[\cos(\sin\theta)]$ (B) $e^{\cos\theta}[\cos(\cos\theta)]$ (C) $e^{\sin\theta}[\sin(\cos\theta)]$ (D) $e^{\sin\theta}[\sin(\sin\theta)]$

10. Let z and ω are two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z + \arg \omega = \pi$, then z equal to -

(A) ω (B) – ω (C) $\bar{\omega}$ (D) – $\bar{\omega}$

11. Number of values of x (real or complex) simultaneously satisfying the system of equations $1 + z + z^2 + z^3 + \ldots\ldots + z^{17} = 0$ and $1 + z + z^2 + z^3 + \ldots\ldots + z^{13} = 0$ is -

(A) 1 (B) 2 (C) 3 (D) 4

12. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$ then the value of $|z_1 + z_2 + z_3|$ is equal to-

(A) 2 (B) 3 (C) 4 (D) 6

13. A point 'z' moves on the curve $|z - 4 - 3i| = 2$ in an argand plane. The maximum and minimum values of |z| are -

(A) 2, 1 (B) 6, 5 (C) 4, 3 (D) 7, 3

14. The set of points on the complex plane such that $z^2 + z + 1$ is real and positive (where $z = x + iy$, $x, y \in R$) is-

(A) Complete real axis only

(B) Complete real axis or all points on the line 2x + 1 = 0

(C) Complete real axis or a line segment joining points $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ & $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ excluding both.

(D) Complete real axis or set of points lying inside the rectangle formed by the lines.

$2x + 1 = 0$; $2x - 1 = 0$; $2y - \sqrt{3} = 0$ & $2y + \sqrt{3} = 0$

15. If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^7$ equals [JEE 98]

(A) 128ω (B) -128ω (C) $128\omega^2$ (D) $-128\omega^2$

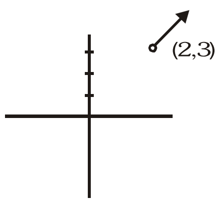
16. If $i=\sqrt{-1}$, then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to : [JEE 99]

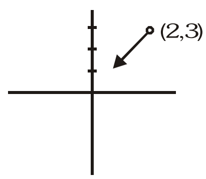
(A) $1-i\sqrt{3}$ (B) $-1+i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

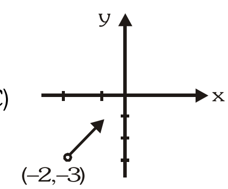
17. The set of points on an Argand diagram which satisfy both $|z| \le 4$ & Arg $z=\frac{\pi}{3}$ are lying on -

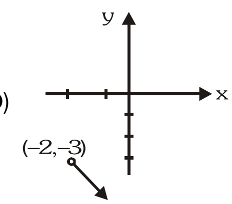
(A) a circle & a line (B) a radius of a circle (C) a sector of a circle (D) an infinite part line

18. If Arg $(z-2-3i)=\frac{\pi}{4}$, then the locus of z is -

(A) 

(B) 

(C) 

(D) 

19. The origin and the roots of the equation $z^2+pz+q=0$ form an equilateral triangle if -

(A) $p^2=2q$ (B) $p^2=q$ (C) $p^2=3q$ (D) $q^2=3p$

20. Points z_1 & z_2 are adjacent vertices of a regular octagon. The vertex z_3 adjacent to z_2 ($z_3 \ne z_1$) can be represented by -

(A) $z_2+\frac{1}{\sqrt{2}}(1\pm i)(z_1+z_2)$ (B) $z_2+\frac{1}{\sqrt{2}}(-1\pm i)(z_1-z_2)$

(C) $z_2+\frac{1}{\sqrt{2}}(-1\pm i)(z_2-z_1)$ (D) none of these

21. $\left[\frac{-1+i\sqrt{3}}{2}\right]^6+\left[\frac{-1-i\sqrt{3}}{2}\right]^6+\left[\frac{-1+i\sqrt{3}}{2}\right]^5+\left[\frac{-1-i\sqrt{3}}{2}\right]^5$ is equal to -

(A) 1 (B) –1 (C) 2 (D) none of these

22. If z and ω are two non-zero complex numbers such that $|z\omega|=1$, and Arg (z) – Arg(ω) = $\pi/2$, then $\bar{z}\,\omega$ is equal to -

(A) 1 (B) –1 (C) i (D) –i

<u>SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)</u>

23. For two complex numbers z_1 and z_2 : $(az_1+b\bar{z}_1)(cz_2+d\bar{z}_2)=(cz_1+d\bar{z}_1)(az_2+b\bar{z}_2)$ if (a, b, c, d $\in$ R) -

(A) $\frac{a}{b}=\frac{c}{d}$ (B) $\frac{a}{d}=\frac{b}{c}$ (C) $|z_1|=|z_2|$ (D) $\arg(z_1)=\arg(z_2)$

24. Which of the following, locii of z on the complex plane represents a pair of straight lines ?

(A) $Re(z^2)=0$ (B) $Im(z^2)=0$ (C) $|z|+z=0$ (D) $|z-1|=|z-i|$

25. If the complex numbers z_1, z_2, z_3 represents vertices of an equilateral triangle such that $|z_1|=|z_2|=|z_3|$, then which of following is correct ?

(A) $z_1+z_2+z_3 \ne 0$ (B) $Re(z_1+z_2+z_3)=0$ (C) $Im(z_1+z_2+z_3)=0$ (D) $z_1+z_2+z_3=0$

26. If S be the set of real values of x satisfying the inequality $1-\log_2\frac{|x+1+2i|-2}{\sqrt{2}-1}\ge 0$, then S contains -

(A) [–3, –1) (B) (–1, 1] (C) [–2, 2] (D) [–3, 1]

27. If amp $(z_1z_2) = 0$ and $|z_1| = |z_2| = 1$, then :-

(A) $z_1 + z_2 = 0$ (B) $z_1z_2 = 1$ (C) $z_1 = \overline{z}_2$ (D) none of these

28. If the vertices of an equilateral triangle are situated at z =0, $z=z_1$, $z=z_2$, then which of the following is/are true -

(A) $|z_1| = |z_2|$ (B) $|z_1 - z_2| = |z_1|$

(C) $|z_1 + z_2| = |z_1| + |z_2|$ (D) $|\arg z_1 - \arg z_2| = \pi/3$

29. Value(s) of $(-i)^{1/3}$ is/are -

(A) $\frac{\sqrt{3}-i}{2}$ (B) $\frac{\sqrt{3}+i}{2}$ (C) $\frac{-\sqrt{3}-i}{2}$ (D) $\frac{-\sqrt{3}+i}{2}$

30. If centre of square ABCD is at z=0. If affix of vertex A is z_1, centroid of triangle ABC is/are -

(A) $\frac{z_1}{3}(\cos \pi + i \sin \pi)$ (B) $4\left[\left(\cos\frac{\pi}{2}\right)-i\left(\sin\frac{\pi}{2}\right)\right]$

(C) $\frac{z_1}{3}\left[\left(\cos\frac{\pi}{2}\right)+i\left(\sin\frac{\pi}{2}\right)\right]$ (D) $\frac{z_1}{3}\left[\left(\cos\frac{\pi}{2}\right)-i\left(\sin\frac{\pi}{2}\right)\right]$

31. If ω is an imaginary cube root of unity, then a root of equation $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+2 \end{vmatrix} = 0$, can be :-

(A) $x = 1$ (B) $x = \omega$ (C) $x = \omega^2$ (D) $x = 0$

CHECK YOUR GRASP — **ANSWER KEY** — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	A	B	B	D	C	C	D	A	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	A	A	D	B	D	C	C	A	C	B
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	A	D	A,D	A,B	B,C,D	A,B	B.C	A,B,D	A,C	C,D
Que.	31									
Ans.	D									

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. On the argand plane, let $\alpha = -2 + 3z$, $\beta = -2 - 3z$ & $|z| = 1$. Then the correct statement is -

(A) α moves on the circle, centre at (–2,0) and radius 3

(B) α & β describe the same locus

(C) α & β move on different circles

(D) $\alpha - \beta$ moves on a circle concentric with $|z|=1$

2. The value of $i^n + i^{-n}$, for $i = \sqrt{-1}$ and $n \in I$ is -

(A) $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$ (B) $\frac{(1+i)^{2n}}{2^n} + \frac{(1-i)^{2n}}{2^n}$ (C) $\frac{(1+i)^{2n}}{2^n} - \frac{2^n}{(1-i)^{2n}}$ (D) $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$

3. The common roots of the equations $z^3 + (1+i)z^2 + (1+i)z + i = 0$, (where $i = \sqrt{-1}$) and $z^{1993} + z^{1994} + 1 = 0$ are - (where ω denotes the complex cube root of unity)

(A) 1 (B) ω (C) ω^2 (D) ω^{981}

4. If $x_r = \text{CiS}\left(\frac{\pi}{2^r}\right)$ for $1 \le r \le n$; r, $n \in N$ then -

(A) $\lim_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = -1$ (B) $\lim_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = 0$ (C) $\lim_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = 1$ (D) $\lim_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = 0$

5. Let z_1, z_2 be two complex numbers represented by points on the circle $|z_1| = 1$ and $|z_2|=2$ respectively, then -

(A) $\max|2z_1 + z_2| = 4$ (B) $\min|z_1 - z_2| = 1$ (C) $\left|z_2 + \frac{1}{z_1}\right| \le 3$ (D) none of these

6. If α, β be any two complex numbers such that $\left|\frac{\alpha - \beta}{1 - \bar{\alpha}\beta}\right| = 1$, then which of the following may be true -

(A) $|\alpha| = 1$ (B) $|\beta| = 1$ (C) $\alpha = e^{i\theta}$, $\theta \in R$ (D) $\beta = e^{i\theta}$, $\theta \in R$

7. Let z, ωz and $z + \omega z$ represent three vertices of ΔABC, where ω is cube root unity, then -

(A) centroid of ΔABC is $\frac{2}{3}(z + \omega z)$ (B) orthocenter of ΔABC is $\frac{2}{3}(z + \omega z)$

(C) ABC is an obtuse angled triangle (D) ABC is an acute angled triangle

8. Which of the following complex numbers lies along the angle bisectors of the line -

$L_1 : z = (1 + 3\lambda) + i(1 + 4\lambda)$

$L_2 : z = (1 + 3\mu) + i(1 - 4\mu)$

(A) $\frac{11}{5} + i$ (B) $11 + 5i$ (C) $1 - \frac{3i}{5}$ (D) $5 - 3i$

9. Let z and ω are two complex numbers such that $|z| \le 1$, $|\omega| \le 1$ and $|z + i\omega| = |z - i\bar{\omega}| = 2$, then z equals -

(A) 1 or i (B) i or –i (C) 1 or –1 (D) i or –1

10. If g(x) and h(x) are two polynomials such that the polynomial $P(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then -

(A) g(1) = h(1) = 0 (B) g(1) = h(1) ≠ 0 (C) g(1) = –h(1) (D) g(1) + h(1) = 0

BRAIN TEASERS — ANSWER KEY — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A,B,D	B,D	B,C	A,D	A,B,C	A,B,C,D	A,C	A,C	C	A,C,D

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

MATCH THE COLUMN

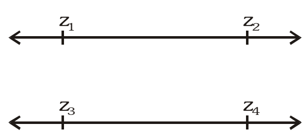
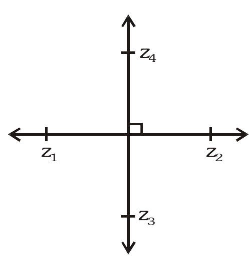
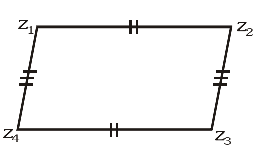
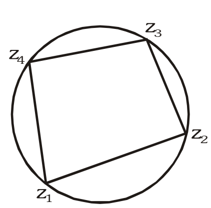
Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1.

	Column-I		Column-I
(A)	If z be the complex number such that $\left\|z+\frac{1}{z}\right\|=2$ then minimum value of $\frac{\|z\|}{\tan\frac{\pi}{8}}$ is	(p)	0
(B)	$\|z\| = 1$ & $z^{2n}+1 \neq 0$ then $\frac{z^n}{z^{2n}+1}-\frac{\overline{z}^n}{\overline{z}^{2n}+1}$ is equal to	(q)	3
(C)	If $8iz^3 + 12z^2 - 18z + 27i = 0$ then $2\|z\| =$	(r)	11
(D)	If z_1, z_2, z_3, z_4 are the roots of equation $z^4 + z^3 + z^2 + z + 1 = 0$, then $\prod_{i=1}^{4}(z_i + 2)$ is	(s)	1

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

2. Match the figure in column-I with corresponding expression -

	Column-I		Column-I
(A)	 two parallel lines	(p)	$\frac{z_4-z_3}{z_2-z_1}+\frac{\overline{z}_4-\overline{z}_3}{\overline{z}_2-\overline{z}_1}=0$
(B)	 two perpendicular lines	(q)	$\frac{z_2-z_1}{z_4-z_3}=\frac{\overline{z}_2-\overline{z}_1}{\overline{z}_4-\overline{z}_3}$
(C)	 a parallelogram	(r)	$\frac{z_4-z_1}{z_2-z_1}\cdot\frac{z_2-z_3}{z_4-z_3}=\frac{\overline{z}_4-\overline{z}_1}{\overline{z}_2-\overline{z}_1}\cdot\frac{\overline{z}_2-\overline{z}_3}{\overline{z}_4-\overline{z}_3}$
(D)		(s)	$z_1 + z_3 = z_2 + z_4$

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : There are exactly two complex numbers which satisfy the complex equations $|z - 4 - 5i| = 4$ and Arg $(z - 3 - 4i) = \frac{\pi}{4}$ simultaneously.

Because

Statement-II : A line cuts the circle in atmost two points.

(A) A (B) B (C) C (D) D

2. Let z_1, z_2, z_3 satisfy $\left|\frac{z+2}{z-1}\right| = 2$ and $z_0 = 2$. Consider least positive arguments wherever required.

Statement–1 : $2\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{z_1 - z_0}{z_2 - z_0}\right)$.

and

Statement–2 : z_1, z_2, z_3 satisfy $|z - z_0| = 2$.

(A) A (B) B (C) C (D) D

3. **Statement-I** : If $z = i + 2i^2 + 3i^3 + \ldots\ldots\ldots\ldots + 32i^{32}$, then z, $\bar{z}$, $-z$ & $-\bar{z}$ forms the vertices of square on argand plane.

Because

Statement-II : z, $\bar{z}$, $-z$, $-\bar{z}$ are situated at the same distance from the origin on argand plane.

(A) A (B) B (C) C (D) D

4. **Statement-I** : If $z_1 = 9 + 5i$ and $z_2 = 3 + 5i$ and if arg $\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$ then $|z - 6 - 8i| = 3\sqrt{2}$

Because

Statement-II : If z lies on circle having z_1 & z_2 as diameter then arg $\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$.

(A) A (B) B (C) C (D) D

5. **Statement-1** : Let z_1, z_2, z_3 be three complex numbers such that $|3z_1 + 1| = |3z_2 + 1| = |3z_3 + 1|$ and $1 + z_1 + z_2 + z_3 = 0$, then z_1, z_2, z_3 will represent vertices of an equilateral triangle on the complex plane.

and

Statement-2 : z_1, z_2, z_3 represent vertices of an equilateral triangle if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1 :

Let z be any complex number. To factorise the expression of the form $z^n - 1$, we consider the equation $z^n = 1$. This equation is solved using De moiver's theorem. Let $1, \alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_{n-1}$ be the roots of this equation, then $z^n - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2)\ldots\ldots\ldots(z - \alpha_{n-1})$ This method can be generalised to factorize any expression of the form $z^n - k^n$.

for example, $z^7 + 1 = \prod_{m=0}^{6}\left(z - \text{C iS}\left(\frac{2m\pi}{7} + \frac{\pi}{7}\right)\right)$

This can be further simplified as

$z^7 + 1 = (z + 1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$ .... (i)

These factorisations are useful in proving different trigonometric identities e.g. in eqaution (i) if we put z = i, then equation (i) becomes

$$(1-i) = (i+1)\left(-2i\cos\frac{\pi}{7}\right)\left(-2i\cos\frac{3\pi}{7}\right)\left(-2i\cos\frac{5\pi}{7}\right)$$

i.e. $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}$

On the basis of above information, answer the following questions :

1. If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5-32) = (z-2)(z^2-pz+4)(z^2-qz+4)$, where p > q, then the value of $p^2 - 2q$ -

(A) 8 (B) 4 (C) –4 (D) –8

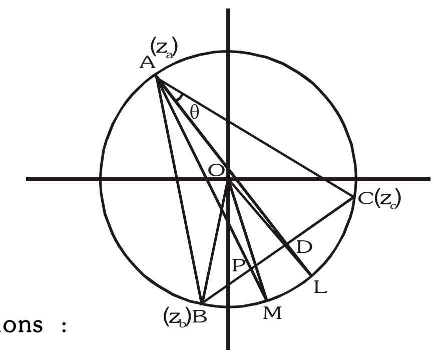
2. By using the factorisation for $z^5 + 1$, the value of $4\sin\frac{\pi}{10}\cos\frac{\pi}{5}$ comes out to be -

(A) 4 (B) 1/4 (C) 1 (D) –1

3. If $(z^{2n+1} - 1) = (z-1)(z^2 - p_1 z + 1)........ (z^2 - p_n z + 1)$ where $n \in N$ & p_1, p_2 .... p_n are real numbers then $p_1 + p_2 + ... + p_n =$

(A) –1 (B) 0 (C) $\tan(\pi/2n)$ (D) none of these

Comprehension # 2 :



In the figure |z| = r is circumcircle of ΔABC.D,E & F are the middle points of the sides BC, CA & AB respectively, AD produced to meet the circle at L. If ∠CAD = θ, AD = x, BD = y and altitude of ΔABC from A meet the circle |z|= r at M, z_a, z_b & z_c are affixes of vertices A, B & C respectively.

On the basis of above information, answer the following questions :

1. Area of the ΔABC is equal to -

(A) xy cos (θ + C) (B) (x + y) sin θ (C) xy sin (θ + C) (D) $\frac{1}{2}$ xy sin (θ + C)

2. Affix of M is -

(A) $2z_b e^{i2B}$ (B) $z_b e^{i(\pi-2B)}$ (C) $z_b e^{iB}$ (D) $2z_b e^{iB}$

3. Affix of L is -

(A) $z_b e^{i(2A-2\theta)}$ (B) $2z_b e^{i(2A-2\theta)}$ (C) $z_b e^{i(A-\theta)}$ (D) $2z_b e^{i(A-\theta)}$

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** EXERCISE-3

- ***Match the Column***

1. (A) → (s), (B) → (p), (C) → (q), (D) → (r)   2. (A) → (q), (B) → (p), (C) → (q, s), (D) → (r)

- ***Assertion & Reason***

1. D 2. A 3. B 4. C 5. B

- ***Comprehension Based Questions***

Comprehension # 1 : 1. A 2. C 3. A

Comprehension # 2 : 1. C 2. B 3. A

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

1. Find the modulus, argument and the principal argument of the complex numbers.

(a) $z = 1 + \cos\frac{10\pi}{9} + i\sin\left(\frac{10\pi}{9}\right)$ (b) $(\tan 1 - i)^2$

(c) $z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$

2. Given that x , y ∈ R , solve : $4x + 3xy + (2xy - 3x)i = 4y - (x^2/2) + (3xy - 2y)i$

3. Let z_1 and z_2 be two complex numbers such that $\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$, find $|z_1|$.

4. If $iz^3 + z^2 - z + i = 0$, then prove that $|z|=1$.

5. If A, B and C are the angle of a triangle $D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$ where $i = \sqrt{-1}$, then find the value of D.

6. For complex numbers z & ω , prove that, $|z|^2\,\omega - |\omega|^2\,z = z - \omega$ if and only if, $z = \omega$ or $z\bar{\omega} = 1$

7. Let z_1, z_2 be complex numbers with $|z_1|=|z_2|=1$, prove that $|z_1 + 1| + |z_2 + 1| + |z_1 z_2 + 1| \geq 2$

8. Interpret the following locii in $z \in C$.

(a) $1 < |z - 2i| < 3$

(b) $\text{Re}\left(\frac{z+2i}{iz+2}\right) \leq 4 \ (z \neq 2i)$

(c) $\text{Arg}(z + i) - \text{Arg}(z - i) = \pi/2$

(d) $\text{Arg}(z - a) = \pi/3$ where $a = 3 + 4i$.

9. Let $A = \{a \in R \mid$ the equation $(1 + 2i)x^3 - 2(3 + i)x^2 + (5 - 4i)x + 2a^2 = 0\}$ has at least one real root. Find the value of $\sum_{a \in A} a^2$.

10. ABCD is a rhombus in the Argand plane. If the affixes of the vertices be z_1, z_2, z_3, z_4 and taken in anti-clockwise sense and $\angle CBA = \pi/3$, show that

(a) $2z_2 = z_1(1 + i\sqrt{3}) + z_3(1 - i\sqrt{3})$ & (b) $2z_4 = z_1(1 - i\sqrt{3}) + z_3(1 + i\sqrt{3})$

11. P is a point on the Argand plane. On the circle with OP as diameter two points Q & R are taken such that ∠POQ = ∠QOR = θ. If 'O' is the origin & P, Q & R are represented by the complex numbers Z_1, Z_2 & Z_3 respectively, show that : $Z_2^2 \cos 2\theta = Z_1 . Z_3 \cos^2\theta$.

12. Let $A \equiv z_1$; $B \equiv z_2$; $C \equiv z_3$ are three complex numbers denoting the vertices of an acuteangled triangle. If the origin 'O' is the orthocentre of the triangle, then prove that $z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$.

13. (a) If ω is an imaginary cube root of unity then prove that :

$(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$..... to 2n factors = $2^{2n.}$

(b) If ω is a complex cube root of unity , find the value of ;

$(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$..... to n factors.

14. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum 3 + 4i and the other two with product 13 + i. Find the value of 'b'.

15. If $x = 1 + i\sqrt{3}$; $y = 1 - i\sqrt{3}$ & $z = 2$, then prove that $x^p + y^p = z^p$ for every prime $p > 3$.

CONCEPTUAL SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(A)

1. (a) Principal Arg $z = -\frac{4\pi}{9}$; $|z| = 2\cos\frac{4\pi}{9}$; Arg $z = 2k\pi - \frac{4\pi}{9}$ $k \in I$

(b) Modulus = $\sec^2 1$, Arg $z = 2n\pi + (2 - \pi)$, Principal Arg $z = (2 - \pi)$

(c) Principal value of Arg $z = -\frac{\pi}{2}$ & $|z| = \frac{3}{2}$, Arg $z = 2n\pi - \frac{\pi}{2}, n \in I$

Principal value of Arg $z = \frac{\pi}{2}$ & $|z| = \frac{2}{3}$, Arg $z = 2n\pi + \frac{\pi}{2}, n \in I$

2. $x = K$, $y = \frac{3K}{2}$ $K \in R$ **3.** 2 **5.** −4

8. (a) The region between the concentric circles with centre at (0 , 2) & radii 1 & 3 units

(b) region outside or on the circle with centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$

(c) semi circle (in the 1st & 4th quadrant) x + y = 1

(d) a ray emanating from the point (3 + 4i) directed away from the origin & having equation $\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$

9. 18 **13.** (b) one if n is even ; $-\omega$ if n is odd **14.** 51

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. (a) Let z = x + iy be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z \mid |z| \le 2\}$ and $B = \{z \mid (1-i)z + (1+i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

(b) For all real numbers x, let the mapping $f(x) = \frac{1}{x-i}$, where $i = \sqrt{-1}$. If there exist real numbers a, b, c and d for which f(a), f(b), f(c) and f(d) form a square on the complex plane. Find the area of the square.

2. If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$; where p , q , r are the moduli of non-zero complex numbers u, v, w respectively, prove that, $\arg \frac{w}{v} = \arg\left(\frac{w-u}{v-u}\right)^2$.

3. For $x \in (0, \pi/2)$ and $\sin x = \frac{1}{3}$, if $\sum_{n=0}^{\infty} \frac{\sin(nx)}{3^n} = \frac{a+b\sqrt{b}}{c}$ then find the value of (a + b + c), where a, b, c are positive integers. (You may use the fact that $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$)

4. If z_1, z_2 are the roots of the equation $az^2 + bz + c = 0$, with a, b, c > 0 ; $2b^2 > 4ac > b^2$; $z_1 \in$ third quadrant ; $z_2 \in$ second quadrant in the argand's plane then, show that

$$\arg\left(\frac{z_1}{z_2}\right) = 2\cos^{-1}\left(\frac{b^2}{4ac}\right)^{1/2}$$

5. If Z_r, r = 1, 2, 3,..... 2m, $m \in N$ are the roots of the equation $Z^{2m} + Z^{2m-1} + Z^{2m-2} + ... + Z + 1 = 0$ then prove that $\sum_{r=1}^{2m} \frac{1}{Z_r - 1} = -m$

6. If $(1 + x)^n = C_0 + C_1x + ...+ C_nx^n$ $(n \in N)$, prove that :

(a) $C_0 + C_4 + C_8 + ... = \frac{1}{2}\left[2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4}\right]$

(b) $C_1 + C_5 + C_9 + ... = \frac{1}{2}\left[2^{n-1} + 2^{n/2}\sin\frac{n\pi}{4}\right]$

(c) $C_2 + C_6 + C_{10} + ... = \frac{1}{2}\left[2^{n-1} - 2^{n/2}\cos\frac{n\pi}{4}\right]$

(d) $C_3 + C_7 + C_{11} + ... = \frac{1}{2}\left[2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right]$

(e) $C_0 + C_3 + C_6 + C_9 + ... = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$

7. Prove that : (a) $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + ... + {}^nC_n \cos (n+1)x = 2^n . \cos^n\frac{x}{2} . \cos\left(\frac{n+2}{2}\right)x$

(b) $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + ... + {}^nC_n \sin (n+1)x = 2^n . \cos^n\frac{x}{2} . \sin\left(\frac{n+2}{2}\right)x$

8. The points A, B, C depict the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. Show that : $(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$

9. Evaluate : $\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$.

10. Let a, b, c be distinct complex numbers such that $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$. Find the value of k.

BRAIN STORMING SUBJECTIVE EXERCISE — ANSWER KEY — EXERCISE-4(B)

1. (a) $\pi - 2$ (b) 1/2	**3.** 41	**9.** $48(1 - i)$	**10.** $-\omega$ or $-\omega^2$

EXERCISE - 05 [A] JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. The inequality $|z - 4| < |z - 2|$ represents the following region [AIEEE-2002]

(1) Re(z) > 0 (2) Re(z) < 0 (3) Re(z) > 2 (4) none of these

2. Let z and ω are two non-zero complex numbers such that $|z| = |\omega|$ and arg z + arg $\omega = \pi$, then z equal to [AIEEE-2002]

(1) ω (2) $-\omega$ (3) $\bar{\omega}$ (4) $-\bar{\omega}$

3. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex, Further, assume that the origin z_3, z_1 and z_2 form an equilateral triangle. then- [AIEEE-2003]

(1) $a^2 = b$ (2) $a^2 = 2b$ (3) $a^2 = 3b$ (4) $a^2 = 4b$

4. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and Arg(z) –Arg(ω) = $\pi/2$, then $\bar{z}\,\omega$ is equal to [AIEEE-2003]

(1) 1 (2) –1 (3) i (4) –i

5. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then [AIEEE-2003]

(1) x = 4n, where n is any positive integer (2) x = 2n, where n is any positive integer

(3) x = 4n + 1, where n is any positive integer (4) x = 2n + 1, where n is any positive integer

6. Let z, w be complex numbers such that $\bar{z} + i\,\bar{w} = 0$ and arg zw = π. Then arg z equals [AIEEE-2004]

(1) $\pi/4$ (2) $\pi/2$ (3) $3\pi/4$ (4) $5\pi/4$

7. If $|z^2 - 1| = |z|^2 + 1$, then z lies on [AIEEE-2004]

(1) the real axis (2) the imaginary axis (3) a circle (4) an ellipse

8. If z = x – iy and $z^{1/3} = p + iq$, then $\frac{\left(\frac{x}{p}+\frac{y}{q}\right)}{(p^2+q^2)}$ is equal to- [AIEEE-2004]

(1) 1 (2) –1 (3) 2 (4) –2

9. If z_1 and z_2 are two non zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$ then arg z_1 – arg z_2 is equal to- [AIEEE-2005]

(1) $-\pi$ (2) $\frac{\pi}{2}$ (3) $-\frac{\pi}{2}$ (4) 0

10. If $w = \frac{z}{z - \frac{1}{3}i}$ and $|w| = 1$ then z lies on [AIEEE-2005]

(1) a circle (2) an ellipse (3) a parabola (4) a straight line

11. If $|z + 4| \le 3$, then the maximum value of $|z + 1|$ is- [AIEEE-2007]

(1) 4 (2) 10 (3) 6 (4) 0

12. The conjugate of a complex number is $\frac{1}{i-1}$, then that complex number is- [AIEEE-2008]

(1) $\frac{-1}{i-1}$ (2) $\frac{1}{i+1}$ (3) $\frac{-1}{i+1}$ (4) $\frac{1}{i-1}$

13. If $\left|Z - \frac{4}{Z}\right| = 2$, then the maximum value of $|Z|$ is equal to :- [AIEEE-2009]

(1) 2 (2) $2 + \sqrt{2}$ (3) $\sqrt{3} + 1$ (4) $\sqrt{5} + 1$

14. The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals :- **[AIEEE-2010]**

(1) 0 (2) 1 (3) 2 (4) ∞

15. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re z = 1, then it is necessary that :- **[AIEEE-2011]**

(1) $|\beta| = 1$ (2) $\beta \in (1, \infty)$ (3) $\beta \in (0, 1)$ (4) $\beta \in (-1, 0)$

16. If $\omega(\neq 1)$ is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals :- **[AIEEE-2011]**

(1) (1, 0) (2) (–1, 1) (3) (0, 1) (4) (1, 1)

17. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies : **[AIEEE-2012]**

(1) on the imaginary axis.

(2) either on the real axis or on a circle passing through the origin.

(3) on a circle with centre at the origin.

(4) either on the real axis or on a circle not passing through the origin.

18. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals **[JEE (Main)-2013]**

(1) $-\theta$ (2) $\frac{\pi}{2} - \theta$ (3) θ (4) $\pi - \theta$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans	4	4	3	4	1	3	2	4	4	4	3	3	4	2	2
Que.	16	17	18												
Ans	4	2	3												

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. (a) If z_1, z_2, z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$ then $|z_1+z_2+z_3|$ is -

(A) equal to 1 (B) less than 1 (C) greater than 3 (D) equal to 3

(b) If arg (z) < 0, then arg (–z) – arg (z) = [JEE 2000 Screening) 1+1M out of 35]

(A) π (B) $-\pi$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

2. (a) The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1-z_3}{z_2-z_3}=\frac{1-i\sqrt{3}}{2}$ are the vertices of a triangle which is -

(A) of area zero (B) right-angled isosceles (C) equilateral (D) obtuse-angled isosceles

(b) Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin. Then n must be of the form

(A) 4k + 1 (B) 4k + 2 (C) 4k + 3 (D) 4k

[JEE 2001 (Screening) 1+1M out of 35]

3. (a) Let $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix}1 & 1 & 1\\ 1 & -1-\omega^2 & \omega^2\\ 1 & \omega^2 & \omega^4\end{vmatrix}$ is - [JEE 02 (Screening) 3M]

(A) 3ω (B) $3\omega(\omega-1)$ (C) $3\omega^2$ (D) $3\omega(1-\omega)$

(b) For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2-3-4i| = 5$, the minimum value of $|z_1 - z_2|$ is [JEE 02 (Screening) 3M]

(A) 0 (B) 2 (C) 7 (D) 17

(c) Let a complex number α, $\alpha \neq 1$, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$ where p,q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$, but not both together.

[JEE 02 (Mains) 5M]

4. If $|z| = 1$ and $\omega=\frac{z-1}{z+1}$ (where $z \neq -1$), then Re (w) equals – [JEE 03 (Screening) 3M]

(A) 0 (B) $-\frac{1}{|z+1|^2}$ (C) $\left|\frac{z}{z+1}\right| \cdot \frac{1}{|z+1|^2}$ (D) $\frac{\sqrt{2}}{|z+1|^2}$

5. If z_1 and z_2 are two complex numbers such that $|z_1| < 1$ and $|z_2| > 1$ then show that $\left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right|<1$

[JEE 03 (mains) 2M out of 60)]

6. Show that there exists no complex number z such that $|z|<\frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$

where $|a_i| < 2$ for i = 1, 2,.......n. [JEE 03 (mains) 2M out of 60)]

7. The least positive value of 'n' for which $(1 + \omega^2)^n = (1 + \omega^4)^n$, where ω is a non real cube root of unity is -

(A) 2 (B) 3 (C) 6 (D) 4

[JEE 04 (screening) 3M]

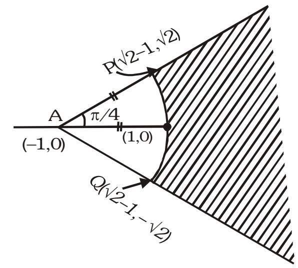
8. Find the centre and radius formed by all the points represented by z = x + i y satisfying the relation $\frac{|z-\alpha|}{|z-\beta|}=K$ $(K \neq 1)$ where α & β are constant complex numbers, given by $\alpha=\alpha_1+i\alpha_2$ & $\beta=\beta_1+i\beta_2$

[JEE 04 (Mains) (2 out of 60)]

9. If a, b, c are integers not all equal and ω is cube root of unity $(\omega \neq 1)$ then the minimum value of $|a + b\omega + c\omega^2|$ is - [JEE 05 (screening) 3M]

(A) 0 (B) 1 (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{2}$

10. Area of shaded region belongs to - [JEE 05 (screening) 3M]



(A) $z : |z + 1| > 2, |\arg (z + 1)| < \pi/4$

(B) $z : |z - 1| > 2, |\arg (z - 1)| < \pi/4$

(C) $z : |z + 1| < 2, |\arg (z + 1)| < \pi/2$

(D) $z : |z - 1| < 2, |\arg (z - 1)| < \pi/2$

11. If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2+\sqrt{3}i$. Find the other vertices of square. [JEE 05 (Mains) 4 out of 60]

12. If $w = \alpha + i\beta$ where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\frac{w - \bar{w}z}{1 - z}$ is purely real, then the set of values of z is - [JEE 06, 3M]

(A) $\{z : |z|=1\}$ (B) $\{z : z = \bar{z}\}$ (C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1, z \neq 1\}$

13. A man walks a distance of 3 units from the origin towards the north-east (N 45 E) direction. From there, he walks a distance of 4 units towards the north-west (N 45 W) direction to reach a point P. Then the position of P in the Argand plane is : [JEE 07, 3M]

(A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)e^{i\pi/4}$ (C) $(4 + 3i)e^{i\pi/4}$ (D) $(3 + 4i)e^{i\pi/4}$

14. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on : [JEE 07, 3M]

(A) a line not passing through the origin (B) $|z| = \sqrt{2}$

(C) the x-axis (D) the y-axis

Comprehension (for 15 to 17) :

Let A, B, C be three sets of complex numbers as defined below [JEE 2008, 4M, –1M]

$$A = \{z : \text{Im}\, z \geq 1\}$$
$$B = \{z : |z - 2 - i| = 3\}$$
$$C = \{z : \text{Re}((1-i)z) = \sqrt{2}\}$$

15. The number of elements in the set $A \cap B \cap C$ is -

(A) 0 (B) 1 (C) 2 (D) ∞

16. Let z be any point in $A \cap B \cap C$. Then $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between -

(A) 25 and 29 (B) 30 and 34 (C) 35 and 39 (D) 40 and 44

17. Let z be any point in $A \cap B \cap C$ and let ω be any point satisfying $|\omega - 2 - i| < 3$. Then, $|z|-|\omega| + 3$ lies between -

(A) –6 and 3 (B) –3 and 6 (C) –6 and 6 (D) –3 and 9

18. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by - [JEE 2008, 3M, –1M]

(A) 6 + 7i (B) –7 + 6i (C) 7 + 6i (D) –6 + 7i

19. Let $z = \cos\theta + i \sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2$ is - [JEE 2009, 3M, –1M]

(A) $\frac{1}{\sin 2°}$ (B) $\frac{1}{3\sin 2°}$ (C) $\frac{1}{2\sin 2°}$ (D) $\frac{1}{4\sin 2°}$

20. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is - [JEE 2009, 3M, –1M]

(A) 48 (B) 32 (C) 40 (D) 80

21. Match the conics in **Column I** with the statements/ expressions in **Column II.** [JEE 2009, 8M]

Column I		Column II
(A) Circle	(P)	The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(B) Parabola	(Q)	Points z in the complex plane satisfying $\lvert z + 2 \rvert - \lvert z - 2 \rvert = \pm 3$
(C) Ellipse	(R)	Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(D) Hyperbola	(S)	The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(T)	Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \lvert z \rvert^2 + 1$

22. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then [JEE 10, 3M]

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$
(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

23. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$ is equal to [JEE 10, 3M]

24. Match the statements in **Column-I** with those in **Column-II.** [JEE 10, 8M]

[**Note** : Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column I	Column II
(A) The set of points z satisfying $\lvert z - i\lvert z\rvert\rvert = \lvert z + i\lvert z\rvert\rvert$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\lvert z + 4\rvert + \lvert z - 4\rvert = 10$ is contained in or equal to	(q) the set of points z satisfying Im z = 0
(C) If $\lvert w\rvert = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(t) the set of points z satisfying $\lvert\text{Im } z\rvert \le 1$
(D) If $\lvert w\rvert = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $\lvert\text{Re } z\rvert \le 2$
	(t) the set of points z satisfying $\lvert z\rvert \le 3$

25. **Comprehension (3 questions together)**

Let a,b and c be three real numbers satisfying

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \ldots(E)$$

(i) If the point P(a,b,c), with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a+b+c$ is

(A) 0 (B) 12 (C) 7 (D) 6

(ii) Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If a = 2 with b and c satisfying (E), then the value of $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is equal to -

(A) –2 (B) 2 (C) 3 (D) –3

(iii) Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is -

(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

[JEE 2011, 3+3+3]

26. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is **[JEE 2011, 4M]**

27. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$a + b + c = x$

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z.$

Then the value of $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is **[JEE 2011, 4M]**

28. Match the statements given in **Column I** with the values given in **Column II**

Column I	Column II
(A) If $\vec{a} = \hat{j}+\sqrt{3}\hat{k}, \vec{b} = -\hat{j}+\sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{\ell n 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r) $\frac{\pi}{3}$
(D) The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for $|z| = 1, z \ne 1$ is given by	(s) π
	(t) $\frac{\pi}{2}$

[JEE 2011, 2+2+2+2M]

29. Match the statements given in **Column I** with the intervals/union of intervals given in **Column II**

Column I	Column II
(A) The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number, } |z| = 1, z \ne \pm 1\right\}$ is	(p) $(-\infty,-1)\cup(1,\infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty,0)\cup(0,\infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is	(r) $[2,\infty)$
(D) If $f(x) = x^{3/2}(3x - 10), x \ge 0$, then $f(x)$ is increasing in	(s) $(-\infty,-1]\cup[1,\infty)$
	(t) $(-\infty,0]\cup[2,\infty)$

[JEE 2011, 2+2+2+2M]

30. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value - **[JEE 2012, 3M, –1M]**

(A) –1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

31. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$ **[JEE(Advanced) 2013, 2M]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

32. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a n n matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n = **[JEE(Advanced) 2013, 3, (–1)]**

(A) 57 (B) 55 (C) 58 (D) 56

33. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in C : \text{Re } z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re } z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2 =$ **[JEE-Advanced 2013, 4, (–1)]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Paragraph for Question 34 and 35

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$ and $S_3 = \{z \in C : \text{Re } z > 0\}$.

34. $\min\limits_{z \in S} |1 - 3i - z| =$ **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

35. Area of S = **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [B]

1. (a) A (b) A **2.** (a) C, (b)D **3.** (a) B ; (b) B **4.** A **7.** B

8. $\frac{\alpha - k^2\beta}{1-k^2}$ & $\left|\frac{1}{k^2-1}\right|\sqrt{|\alpha - k^2\beta|^2 - (k^2|\beta|^2 - |\alpha|^2)(k^2-1)}$ **9.** B **10.** A

11. $(-\sqrt{3}\ i)$, $(1-\sqrt{3}) + i$ and $(1+\sqrt{3}) - i$ **12.** D **13.** D **14.** D **15.** B

16. C **17.** D **18.** D **19.** D **20.** A **21.** A → (P) ; B → (S, T) ; C → (R) ; D → (Q, S)

22. A,C,D **23.** 1 **24.** (A) → (q,r), (B) → (p), (C) → (p,s,t), (D) → (q,r,s,t) **25.** **(i)** D, **(ii)** A, **(iii)** B

26. 5 **27.** Bonus **28.** (A) → (q); (B) → (p) or (p, q, r, s, t); (C) → (s); (D) → (t)

29. (A) → (s); (B) → (t); (C) → (r); (D) → (r) **30.** D **31.** C **32.** B,C,D **33.** C,D

34. C **35.** B